Enveloping Implies Global Stability

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Abstract

Some of the simplest models of population growth are one dimensional nonlinear difference equations. While such models can display wild behavior including chaos, the standard biological models have the interesting property that they display global stability if they display local stability. Various researchers have sought a simple explanation for this agreement of local and global stability. Here, we show that **enveloping** by a linear fractional function is sufficient for global stability. We also show that for seven standard biological models local stability implies enveloping and hence global stability. We derive two methods to demonstrate enveloping and show that these methods can easily be applied to the seven example models.

1 Introduction

Simple population growth models have a pleasant property, they display global convergence if they have local convergence. This fact was established for a number of models by Fisher et al [9, 10] who constructed an explicit Lyapunov function for each model they studied. Since then a number of workers have created a variety of sufficient conditions to demonstrate global stability. [21, 19, 4, 2, 1, 3] Each of these methods suffer from the difficulty that either the method does not apply to one of the commonly used models or the method is computationally difficult to apply.

In this paper, we describe a simple condition which is satisfied by all the commonly used simple population models, and we show that for these models the computation for the method is not difficult. Our simple condition is that the population models are *enveloped* by *linear fractional functions*. No single linear fractional serves for all models. Instead the linear fractionals depend on a single parameter which must be adjusted for the particular model. In some cases, this parameter will also change depending on the parameters of the model. This parameter dependence may be why this simple condition has not been discovered before.

Our pleasure with this result is not solely mathematical. There is also a psychological component. We suspect that the original creators of these models were good biologists and not sophisticated mathematicians. If the similarity among these models required deep and complicated mathematics, we would feel that we had not captured the simple vision of the original modelers. We will argue that the usual way of writing these models suggests an implicit constraint that will force enveloping by a linear fractional.

2 Background and Definitions

In the most general sense, we want to study difference equations of the form

$$x_{t+1} = f(x_t)$$

but with this degree of generality, little can be said. If we require that f is a function which is defined for all values of x, then given an initial condition x_0, we can show that there is a unique solution to the difference equation, that is, x_t traces out a well-defined trajectory. To obtain stronger results, we will assume that f is continuous and has as many continuous derivatives as necessary. As we will see in the examples, we will assume even more structure for a population model. Intuitively, if there is no population now, there will be no population later. If the population is small, we expect it to be growing. If the population is large, we expect it to be decreasing. These ideas suggest that there should be an *equilibrium point* where the population size will remain constant. We expect the function f to be *single-humped*, that is, f should rise to a maximum and then decrease. For some models, f will go to 0 for some finite x, but for other models f will continually decrease toward 0.

We want to know what will happen to x_t for large values of t. Clearly we expect that if x_0 is near $\overline{x}$ then x_t will overshoot and undershoot $\overline{x}$. Possibly this oscillation will be sustained, or possibly x_t will settle down at $\overline{x}$. The next definitions codify these ideas. A population model is ***globally stable*** if and only if for all x_0 such that $f(x_0) > 0$ we have

$$\lim_{t\to\infty} x_t = \overline{x}$$

where $\overline{x}$ is the unique equilibrium point of $x_{t+1} = f(x_t)$. A population model is ***locally stable*** if and only if for every small enough neighborhood of $\overline{x}$ if x_0 is in this neighborhood, then x_t is in this neighborhood for all t, and

$$\lim_{t\to\infty} x_t = \overline{x}.$$

How can we decide if a model has one of these properties? The following well-known theorem gives one answer.

Theorem 1. *If $f(x)$ is differentiable then, a population model is locally stable if $|f'(\overline{x})| < 1$, and if the model is locally stable then $|f'(\overline{x})| \leq 1$.*

For global stability, a slight modification of a very general theorem of Sarkovskii [20] gives:

Theorem 2. *A continuous population model is globally stable iff it has no cycle of period 2. (That is, there is no point except $\overline{x}$ such that $f(f(x)) = x$.)*

This theorem has been noted by Cull[1] and Rosenkranz[19].

Unfortunately, this global stability condition may be difficult to test. Further, there is no obvious connection between the local and global stability conditions.

Various authors have demonstrated global stability for some population models. Fisher *et al* [9] and Goh [10] used Lyapunov functions[13] to show global stability. This technique suffers from the drawbacks that a different Lyapunov function is needed for each model and that there is no systematic method to find these functions. Singer [21] used the negativity of the Schwarzian to show global stability. This technique does not cover all the models we will consider, and it even requires modification to cover all the models it was claimed to cover. Rosenkranz [19] noted that no period 2 was implied by $|f'(x)f'(f(x))| < 1$ and showed that this condition held for a population genetics model. This condition seems to be difficult to test for the models we will consider. Cull [1, 2, 4, 3] developed two conditions **A** and **B** and showed that each of the models we will consider satisfied at least one of these conditions. These conditions used the the first through third derivatives and so were difficult to apply. Also, as Hwang [12] pointed out these conditions required continuous differentiability. All of these methods are relatively mathematically sophisticated, and so it is not clear how biological modelers could intuitively see that these conditions were satisfied.

If we return to the condition for local stability, we see that it says if for x slightly less than 1, $f(x)$ is below a straight line with slope -1, and if for x slightly greater than 1, $f(x)$ is above the same straight line, then the model is locally stable. If we consider the model

$$x_{t+1} = x_t e^{2(1-x_t)},$$

we can see that the local stability bounding line is $2 - x$. Somewhat suprisingly, this line is an upper bound on $f(x)$ for all x in $[0, 1)$ and a lower bound for all $x > 1$. Since $2 - (2 - x) = x$, the bounding by this line can be used to argue that for this model there are no points of period 2, and hence the model is globally stable. From this example, we abstract the following definition. A function $\phi(x)$ ***envelops*** a function $f(x)$ if and only if

- $\phi(x) > f(x)$ for $x \in (0, 1)$
- $\phi(x) < f(x)$ for $x > 1$ such that $\phi(x) > 0$ and $f(x) > 0$

We will use the notation $\phi(x) \bowtie f(x)$ to symbolize this enveloping.

As we will see, our example population models have one or more parameters, and a model with one choice of parameters will envelop the same model with a different choice of parameters. For example, the function $xe^{2(1-x)}$ envelops all the functions of the form $xe^{r(1-x)}$ for $r \in (0, 2)$.

While a straight line was sufficient to envelop $xe^{2(1-x)}$, a straight line fails to envelop the closely related function $x[1 + 2(1 - x)]$. To get a more general enveloping function, we consider the ratio of two linear functions and assume that the ratio is 1 when $x = 1$ and the derivative of this function is -1 when $x = 1$, which gives the following definition.

A ***linear fractional function*** is a function of the form

$$\phi(x) = \frac{1 - \alpha x}{\alpha - (2\alpha - 1)x} \qquad \text{where } \alpha \in [0, 1) \ .$$

These functions have the properties

- $\phi(1) = 1$
- $\phi'(1) = -1$
- $\phi(\phi(x)) = x$
- $\phi'(x) < 0$.

The shape of our linear fractional functions changes markedly as α varies. For $\alpha = 0$, $\phi(x) = 1/x$, which has a pole at $x = 0$, and decreases with an always positive second derivative. For $\alpha \in (0, 1/2)$, $\phi(x)$ starts (for $x = 0$) at $1/\alpha$ and decreases with a positive second derivative. For $\alpha = 1/2$, $\phi(x) = 2 - x$, which starts at 2 and decreases to 0 with a zero second derivative. For $\alpha \in (1/2, 1)$, $\phi(x)$ starts at $1/\alpha$, decreases with a negative second derivative, and hits 0 at $1/\alpha$ which is greater than 1. We are only interested in these functions when $x > 0$ and $\phi(x) > 0$, so we do not care about the pole in these linear fractionals because the pole occurs outside the area of interest. The following figure shows the three different shapes of linear fractional functions.

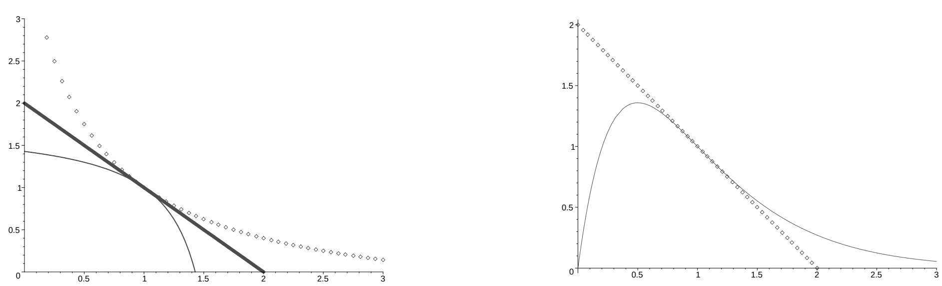

Figure 1: (a)Three types of linear fractionals. Dotted line $\alpha = 1/4$. Heavy line $\alpha = 1/2$. Light line $\alpha = .7$. (b) Model I is enveloped by the straight line $2 - x$ which is the linear fractional with $\alpha = 1/2$.

3 Theorems

We are now in a position to prove the necessary theorems. In what follows, we will assume that our model is $x_{t+1} = f(x_t)$, and that the model has been normalized so that the equilibrium point is 1, that is $f(1) = 1$. We will use the notation $f^{(k)}(x)$ to mean that the function f has been applied k times to x. This notation can be recursively defined by $f^{(0)}(x) = x$ and $f^{(i)}(x) = f(f^{(i-1)}(x))$ for $i \geq 1$.

Theorem 3. *Let $\phi(x)$ be a monotone decreasing function which is positive on $(0, x_-)$ and so that $\phi(\phi(x)) = x$. Assume that $f(x)$ is a continuous function such that:*

- $\phi(x) > f(x)$ *on* $(0, 1)$
- $\phi(x) < f(x)$ *on* $(1, x_-)$
- $f(x) > x$ *on* $(0, 1)$
- $f(x) < x$ *on* $(1, \infty)$
- $f(x) > 0$ *on* $(1, x_\infty)$

then for all $x \in (0, x_\infty)$, $\lim_{k\to\infty} f^{(k)}(x) = 1$.

Proof. From Sarkovskii's theorem, it suffices to show that $f(x)$ has no cycle of period 2. We show that $f(f(x)) > x$ for $x \in (0, 1)$. If $f(f(x)) > 1$ then $f(f(x)) > x$. If $f(f(x)) < 1$ and $f(x) < 1$ then $f(f(x)) > f(x) > x$. If $f(f(x)) < 1$ and $f(x) > 1$, $\phi(f(x)) < f(f(x))$ and $x_- > \phi(x) > f(x)$, and since $\phi(x)$ is decreasing and self inverse $x = \phi(\phi(x)) < \phi(f(x)) < f(f(x))$. A similar argument shows that $x > f(f(x))$ for $x > 1$. (Even if $f(x) > 1$, $f(x) < x_-$ because $x_- > \phi(x)$.) Even though Sarkovskii's theorem assumes a closed interval, we are showing that there are no cycles in an open interval, and so none within the closed intervals inside the open interval. Further our assumptions on $f(x)$ allow us to argue that there is a small ε so that $f^{(k)}(x)$ will eventually fall into the the closed interval $[\varepsilon, \phi(\varepsilon)]$. □

A slight recasting of the above argument gives:

Corollary 1. *If $f_1(x)$ is enveloped by $f_2(x)$, and $f_2(x)$ is globally stable, then $f_1(x)$ is globally stable.*

Corollary 2. *If $f(x)$ is enveloped by a linear fractional function then $f(x)$ is globally stable.*

A function $h(z)$ is ***doubly positive*** iff

1. $h(z)$ has a power series $\sum_{i=0}^{\infty} h_i z^i$
2. $h_0 = 1$, $h_1 = 2$
3. For all $n \geq 1 \quad h_n \geq h_{n+1}$
4. For all $n \geq 2 \quad h_n - 2h_{n+1} + h_{n+2} \geq 0$

Theorem 4. *Let $x_{t+1} = f(x_t)$ where $f(x) = xh(1-x)$ and $h(z)$ is doubly positive, then $f(x)$ is enveloped by the linear fractional function*

$$\phi(x) = \frac{1-\alpha x}{\alpha + (1-2\alpha)x}$$

where $\alpha = \frac{3-h_2}{4-h_2} \geq \frac{1}{2}$ and the model $x_{t+1} = f(x_t)$ is globally stable.

Proof. Recasting in terms of $z = 1-x$ we want to show that $\phi(z) - (1-z)h(z) > 0$ for $z \in (0,1)$ and $\phi(z) - (1-z)h(z) < 0$ for $z \in (-\frac{1-\alpha}{\alpha}, 0)$ where $\phi(z) = \frac{1+\beta z}{1-(1-\beta)z}$ and $\beta = \frac{\alpha}{1-\alpha}$. Assuming that $h(z)$ has a power series, the function we want to bound can be written as:

$$\begin{array}{lllll} 1 & +\beta z & & & \\ -h_0 & -h_1 z & -h_2 z^2 & -h_3 z^3 & -\cdots \\ & +(2-\beta)h_0 z & +(2-\beta)h_1 z^2 & +(2-\beta)h_2 z^3 & +\cdots \\ & & -(1-\beta)h_0 z^2 & -(1-\beta)h_1 z^3 & -\cdots \end{array}$$

By the assumption on h_0 and h_1, the coefficients of z^0 and z^1 vanish. By choosing $\beta = 3 - h_2$ the coefficient of z^2 vanishes. The succeeding coefficients can be written as

$$(\beta - 1)[h_n - h_{n+1}] + [h_{n+1} - h_{n+2}]$$

with $n \geq 1$. By assumption $\beta = 3-h_2 \geq 3-h_1 = 1$. So assuming that $h_n \geq h_{n+1}$ makes all these coefficients nonnegative, and for the power series to converge at least one of these inequalities must be strict, and hence $\phi(z) - (1-z)h(z) > 0$ for $z \in (0,1)$. We have shown that the function has the form $z^3 p(z)$, so to show that it is *negative* on $(-1/\beta, 0)$, which will follow if $p(z)$ is positive on $(-1/\beta, 0)$ and this will follow if $p_n - \frac{1}{\beta}p_{n+1} \geq 0$ where p_n and p_{n+1} are the n^{th} and $n+1^{st}$ coefficients of $p(z)$. From above, this is

$$(\beta - 1)[h_n - h_{n+1}] + \frac{1}{\beta}[h_{n+1} - 2h_{n+2} + h_{n+3}] \geq 0$$

which will be nonnegative by the assumptions, and at least one inequality will be positive if the power series converges. □

While this doubly positive condition will be sufficient for a number of models, it is not sufficient for all the examples because, in particular, β will be less than 1 for some of the models. The following observation will be useful in many cases.

Observation 1. *Let $\phi(x) = A(x)/B(x)$, $f(x) = C(x)/D(x)$ and $G(x) = A(x)D(x) - B(x)C(x)$. If $G(1) = 0$, $G'(1) = 0$, and $G''(x) > 0$ on $(0,1)$ and $G''(x) < 0$ for $x > 1$, then $\phi(x)$ envelops $f(x)$. (We are implicitly assuming that A, B, C, D are all positive, and all functions are twice continuously differentiable.)*

Proof. Obviously, if $G'(1) = 0$ and $G''(x) > 0$ on $(0, 1)$ then $G'(x) < 0$ on $(0, 1)$. Also, if $G''(x) < 0$ for $x > 1$, $G'(x) < 0$ for $x > 1$. But then $G(x)$ is always decreasing, and since $G(1) = 0$, $G(x) > 0$ for $x < 1$ and $G(x) < 0$ for $x > 1$. Rewriting this result shows that $\phi(x)$ envelops $f(x)$. □

If convenient we can switch to the variable $z = 1-x$, and, of course, $G''(x) = G''(z)$. So if $G''(z) = zp(z)$ where $p(z)$ is strictly positive then $\phi(x)$ envelops $f(x)$.

4 Simple Models of Population Growth

4.1 Model I

The model $x_{t+1} = x_t e^{r(1-x_t)}$ is widely used (see, for example [14, 15, 18]). Our first observation is that $0 < r \leq 2$ is the necessary condition for local stability. It is easy to show that this model with $0 < r < 2$ is enveloped by this model with $r = 2$. As we showed earlier, this model with $r = 2$ is enveloped by $\phi(x) = 2-x$ and hence local and global stability coincide.

It is also easy to check that the doubly positive condition holds for this model. Specifically,

$$h(z) = e^{2z} = 1 + z + \frac{2^2 z^2}{2!} + \frac{2^3 z^3}{3!} + \cdots$$

and $h_0 = 1$, and $h_1 = 2$, and

$$h_n - h_{n+1} = \frac{2^n}{(n+1)!}[n + 1 - 2] = \frac{2^n (n-1)}{(n+1)!} \geq 0$$

for $n \geq 1$ and

$$h_n - 2h_{n+1} + h_{n+2} = \frac{2^n}{(n+2)!}[n^2 - n - 2] \geq 0$$

for $n \geq 2$.

4.2 Model II

The model $x_{t+1} = x_t[1+r(1-x_t)]$ is widely used [22] and is sometimes considered to be a truncation of Model I. As for Model I the necessary condition for local stability is $0 < r \leq 2$, and like Model I it is easy to show that this model with $0 < r < 2$ is enveloped by this model with $r = 2$. Unlike Model I, this model is not enveloped by a straight line. But the doubly positive condition holds. Specifically, $h(z) = 1 + 2z$, so

$$h_n - h_{n+1} = \left\{ \begin{array}{ll} 2-0 & n = 1 \\ 0-0 & n > 1 \end{array} \right\} \geq 0$$

and

$$h_n - 2h_{n+1} + h_{n+2} = \left\{ \begin{array}{ll} 2 & n = 1 \\ 0 & n > 1 \end{array} \right\} \geq 0.$$

Since $h_2 = 0$, the enveloping function has $\alpha = \frac{3}{4}$ and is

$$\phi(x) = \frac{4-3x}{3-2x}.$$

In this simple example, it's easy to check that the enveloping condition is equivalent to $(1-x)^3$ having a single change of sign which occurs at $x = 1$.

4.3 Model III

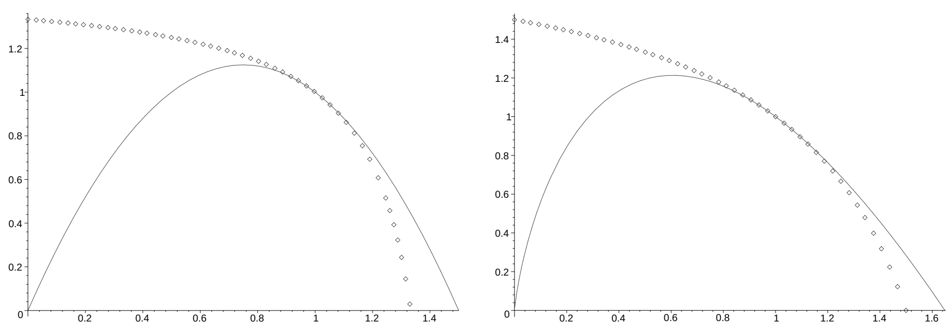

Figure 2: (a) The quadratic map (Model II) is enveloped by $(4-3x)/(3-2x)$. (b) Model III $f(x) = x[1 - 2\ln x]$ enveloped by $(3-2x)/(2-x)$.

The model $x_{t+1} = x_t[1 - r \ln x_t]$ is attributed to Gompertz and studied by Nobile *et al*[16]. As with the preceding two models $0 < r \leq 2$ is the necessary condition for local stability, the model with $r = 2$ envelops the model with $0 < r < 2$, and the doubly positive condition holds. Specifically,

$$h(z) = 1 - 2\ln(1-z) = 1 + 2z + \frac{2z^2}{2} + \cdots + \frac{2z^n}{n} + \cdots$$

and $h_n - h_{n+1} = \frac{2}{n(n+1)} > 0$ for $n \geq 1$ and $h_n - 2h_{n+1} + h_{n+2} = \frac{4}{n(n+1)(n+2)} > 0$ for $n \geq 1$. Since $h_2 = 1$, the enveloping function has $\alpha = 2/3$ and is $\phi(x) = \frac{3-2x}{2-x}$.

4.4 Model IV

Model IV is

$$x_{t+1} = x_t(\frac{1}{b + cx_t} - d).$$

from [24]. This model differs from the previous three in that there are two parameters, b and d, remaining after the carrying capacity has been normalized to 1.

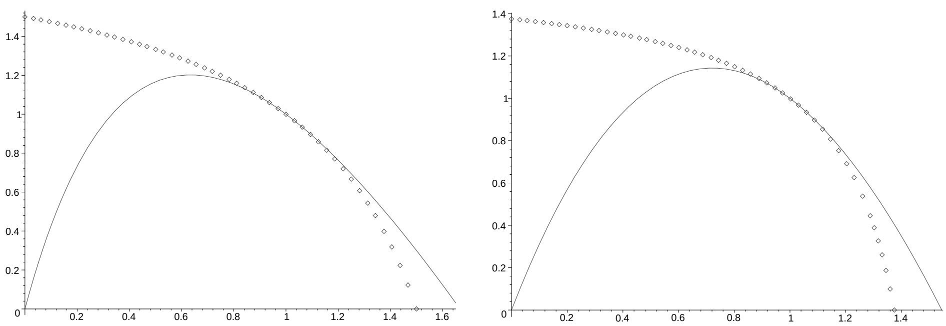

Figure 3: Two examples of model IV. The model with $d = 3$ is enveloped by $(3-2x)/(2-x)$. With $d = 11$ the model is enveloped by $(11 - 8x)/(8 - 5x)$.

The necessary condition for local stability gives

$$\frac{d-1}{(d+1)^2} \leq b < \frac{1}{d+1}.$$

To avoid a pole for $x > 0$, we also assume, $d > 1$. It is easy to check that this model with $b = \frac{d-1}{(d+1)^2}$ envelops this model with larger values of b. With these assumptions

$$f(x) = x[\frac{(d+1)^2}{d-1+2x} - d]$$

and

$$h(z) = \frac{d+1}{1 - \frac{2}{d+1}z} - d.$$

Since $d > 1$,

$$h(z) = 1 + 2z + \frac{2^2}{d+1}z^2 + \frac{2^3}{(d+1)^2}z^3 + \cdots$$

and

$$h_n = \frac{2^n}{(d+1)^{n-1}} \text{ for } n \geq 1.$$

So,

$$h_n - h_{n+1} = \frac{2^n}{(d+1)^n}(d-1) > 0$$

and

$$h_n - 2h_{n+1} + h_{n+2} = \frac{2^n(d-1)^2}{(d+1)^n} > 0.$$

The enveloping function is

$$\phi(x) = \frac{4d - (3d-1)x}{3d - 1 + 2(1-d)x}$$

and has

$$\alpha = \frac{3d-1}{4d} > \frac{1}{2}.$$

We note that $\phi(x)$ has a pole, but $\phi(x)$ goes to zero before the pole, so we can simply ignore the pole. Of course, we only need $\phi(x)$ to bound $f(x)$ on the interval $(0, \frac{4d}{3d-1})$ where $\phi(x)$ is positive.

4.5 Model V

Model V has

$$f(x) = \frac{(1+ae^b)x}{1+ae^{bx}}$$

and comes from Pennycuick *et al*[17]. This and the following two model are more complicated than the previous models because we have to consider different enveloping functions for different parameter ranges.

For $b \le 2$, $xe^{b(1-x)}$ envelops $f(x)$ because $e^{b(1-x)} + ae^{bx} \bowtie 1 + ae^b$ since $e^{b(1-x)} \bowtie 1$ for $b > 0$. (Here we are using the notation $g(x) \bowtie h(x)$ to mean $g(x) > h(x)$ for $x \in (0,1)$ and $g(x) < h(x)$ for $x > 1$ and still in the range of interest.) But $xe^{b(1-x)}$ is just Model I, and as we showed it is enveloped by $2-x$. So Model V is globally stable for $b \le 2$. Of course, the inequality still holds for $b > 2$, but since Model I is *not* stable for $b > 2$, the inequality does not help in establishing the stability of Model V.

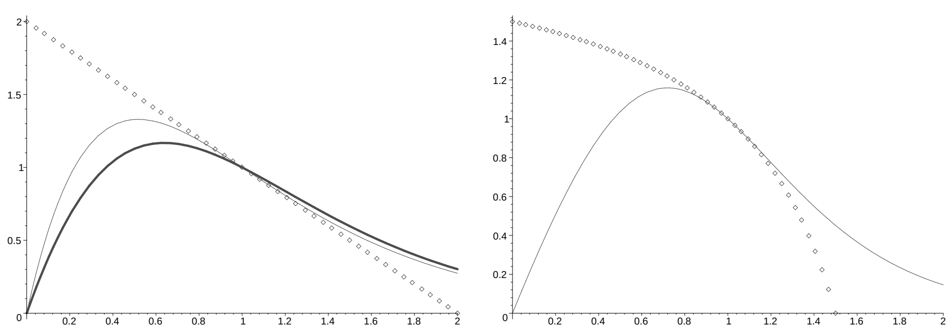

Figure 4: Two examples of model V. The model with $b \le 2$ is enveloped by $(2-x)$. (The curve becomes steeper as a is increased.) With $b = 3$ the model is enveloped by $(3-2x)/(2-x)$.

For this model we assume that $a > 0$ and $b > 0$. The necessary condition for local stability gives $a(b-2)e^b \le 2$. It is easy to show that this model with larger values of a envelops this model with smaller values of a. Letting $ae^b = \frac{2}{b-2}$ and using $z = 1-x$ we have

$$f(z) = \frac{b(1-z)}{(b-2) + 2e^{-bz}}.$$

The enveloping linear fractional is

$$\phi(x) = \frac{b-(b-1)x}{(b-1)-(b-2)x}$$

or converting to $z = 1 - x$,

$$\phi(z) = \frac{1+(b-1)z}{1+(b-2)z}.$$

Following the technique of the Observation, we have,

$$G(z) = (b-2)+2e^{-bz}+(b-1)(b-2)z+2(b-1)ze^{-bz}-b(1-z)-b(b-2)z(1-z).$$

It is easy to check that $G(0) = G'(0) = 0$. Finally,

$$G''(z) = z\{2b(b-2)\left(\frac{1-e^{-bz}}{z}\right) + 2(b-1)b^2e^{-bz}\}.$$

Clearly, $\frac{1-e^{-bz}}{z}$ is positive for all $z \neq 0$ and since $b > 2$, the first term in {-brackets is positive. Of course, the second term is also positive. So by the Observation, Model V is enveloped as claimed.

4.6 Model VI

Model VI is from Hassel [11] and has

$$f(x) = \frac{(1+a)^b x}{(1+ax)^b} \qquad \text{with } a > 0,\ b > 0.$$

There are two cases to consider $0 < b \leq 2$ and $b > 2$. The enveloping function for $b \leq 2$ is $\phi(x) = 1/x$. Cross multiplication shows that we want $(1+ax)^b \bowtie (1+a)^b x^2$. Taking b^{th} roots and rearranging shows that we want $1-x+ax(1-x^{\frac{2-b}{b}}) \bowtie 0$. Clearly, each of the two terms is positive (nonnegative) below 1 and negative (nonpositive) above 1, and so enveloping is established.

The local stability condition implies that $a(b-2) \leq 2$. It is also is easy to show that this model with smaller values of a is enveloped by this model with larger values of a. So if $b > 2$, we can use $a = \frac{2}{b-2}$ or equivalently $b = \frac{2+2a}{2}$ to simplify formulas. Cross multiplication gives

$$G(x) = 2(b-1)(1+ax)^b - (b-2)(1+ax)^b - (b-2)(1+a)^b x - 2(1+a)^b x^2.$$

Simplification and multiplication by $a/2$ gives

$$\begin{aligned} G(x) &\equiv (2+a)(1+ax)^{b-1} - x(1+ax)^{b-1} - (1+a)^b x \qquad \text{and} \\ G''(x) &\equiv (2+a)^2 a(b-2)(1+ax)^{b-3} - a(b-1)(1+ax)^{b-2} \\ &\quad -(2+a)(1+ax)^{b-2} - (2+a)a(b-2)x(1+ax)^{b-3}. \end{aligned}$$

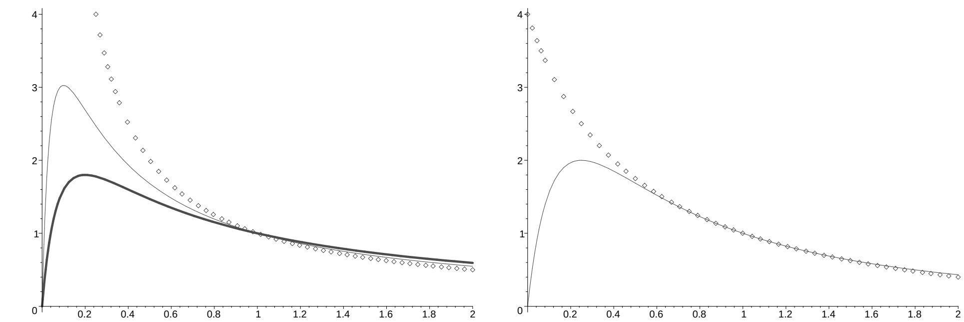

Figure 5: Two examples of model IV. The model with $b \leq 2$ is enveloped by $1/x$. (The curve becomes steeper as a is increased.) With $b = 3$ the model is enveloped by $27x/(1+2x)^3$.

Dividing by $(1+ax)^{b-3}$ and simplifying gives

$$G''(x) \equiv (2+a)^2 2 - 2(2+a)(1+ax) - 2(2+a)x \equiv 2(2+a)(1+a)(1-x).$$

So $G''(x) \bowtie 0$ and enveloping is established. (In this argument we use $\equiv$ to indicate that two quantaties have the same sign, but not necessarily the same value.)

4.7 Model VII

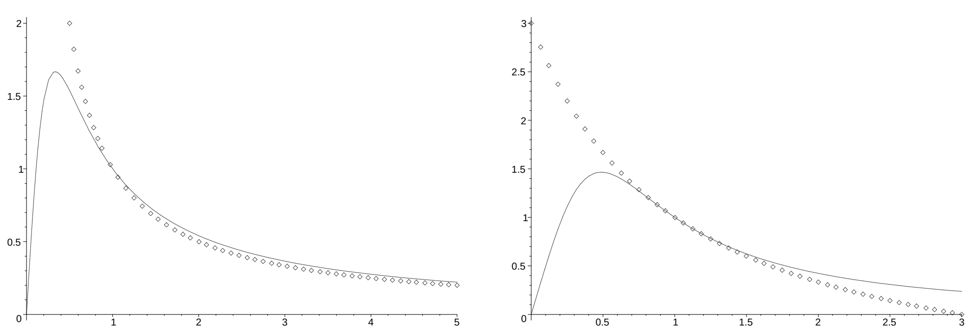

Figure 6: Different linear fractionals envelop Model VII with different parameter values. The fractional $1/x$ is used for $c = 2$. the fractional $(3-x)/(1+x)$ is used with $c = 2.5$, $r = 5$.

Model VII is due to Maynard Smith [23] and has

$$f(x) = \frac{rx}{1+(r-1)x^c}.$$

This seems to be the hardest to analyze model in our set of examples. For example, this model does not satisfy the Schwarzian derivative condition or Cull's condition **A**. Even for our enveloping analysis, we will need to consider this model as three subcases.

Similar to previous models, local stability implies $r(c-2) \leq c$, and it is easy to show that this model with smaller values of r is enveloped by this model with larger values of r.

We first consider the situation when $c \in (0, 2]$. Here, local stability does not place an upper bound on r. Of course, we assume $r > 1$ for this to be a population model. The enveloping function here is $\phi(x) = 1/x$, that is the linear fractional with $\alpha = 0$. Cross multiplication shows that we need

$$1 + (r-1)x^c - rx^2 \bowtie 0$$

for enveloping. Rewriting this gives $1-x^c+rx^c(1-x^{2-c}) \bowtie 0$. Clearly, $1-x^c > 0$ and $1 - x^{2-c} \geq 0$ for $1 > x$ and $2 \geq c$, and $1 - x^c < 0$ and $1 - x^{2-c} \leq 0$ for $1 < x$ and $2 \geq c$, so enveloping is established.

For $c > 2$, we use $r = \frac{c}{c-2}$, and show that

$$\phi(x) = \frac{c-1-(c-2)x}{c-2-(c-3)x}$$

is the enveloping function. As before, we calculate

$$\begin{aligned}
G(x) &= 2x^c[(c-1)-(c-2)x] + (c-1)(c-2) - 2(c-1)(c-2)x + c(c-3)x^2 \\
G'(x) &= 2x^{c-1}[c(c-1)-(c+1)(c-2)x] - 2(c-1)(c-2) + 2c(c-3)x \\
G''(x) &= 2x^{c-2}[c(c-1)^2 - c(c+1)(c-2)x] + 2c(c-3) \\
&= 2c\{(c-1)^2x^{c-2}[1-x] + (c-3)[1-x^{c-1}]\}.
\end{aligned}$$

So for $c \geq 3$, $G''(x) \bowtie 0$ and $\phi(x)$ envelops $f(x)$.

We are left with the case when $c \in (2, 3)$.

$$\frac{G''(x)}{2c} = -(c+1)(c-2)x^{c-1} + (c-1)^2x^{c-2} - (c-3)$$

and so, $G''(x)$ has two positive real roots. One of these is, of course, the root at $x = 1$. Now, taking another derivative, $G''(x)$ is clearly decreasing at $x = 1$, and hence the other root occurs for some $x < 1$. Since $G''(0) < 0$, G'' will start out negative, become positive, and then become negative for all $x > 1$. But now consider $G'(x)$. $G'(0) < 0$ and so while G'' is negative, G' will become more negative, and when G'' becomes positive, G' will increase from a negative value up to 0 at $x = 1$, and then since $G'' < 0$, G' will decrease and stay negative. Hence G which starts positive will decrease through 0 at $x = 1$ and continue decreasing. So, $G(x) \bowtie 0$ and $\phi(x)$ does envelop $f(x)$.

5 Enveloping is Only Sufficient

Here we want to give a simple model which has global stability, but cannot be enveloped by any linear fractional. Define $f(x)$ by

$$f(x) = \begin{cases} 6x & 0 \leq x < 1/2 \\ 7 - 8x & 1/2 \leq x < 3/4 \\ 1 & 3/4 \leq x. \end{cases}$$

then $x_{t+1} = f(x_t)$ has $x = 1$ as its globally stable equilibrium point because if $x_t \geq 1$ then $x_{t+1} = 1$, for $x_t \in [1/2, 1)$, $x_{t+1} > 1$ and $x_{t+2} = 1$, and for $x_t \in (0, 1/2)$, the subsequent iterates grow by multiples of 6 and eventuslly surpass 1/2. This $f(x)$ cannot be enveloped by a linear fractional because $f(1/2) = 3$ which implies that the linear fractional would have $\alpha \leq -1$ and hence have a pole in $(0, 1)$ and thus it could not envelop a positive function. We note that with $\alpha = -1$, $\phi(x)$ would have a pole at $1/3$ and could be used to show that $x = 1$ is globally stable for all $x > 1/3$.

6 Conclusion

Enveloping is a simple technique to demonstrate global stability for some one-dimensional difference equations. Enveloping was introduced by Cull and Chaffee [7, 6, 8]. We demonstrated that the usual population models can be enveloped by linear fractional functions. Such enveloping seems to capture the idea of *simple* function in that a "free-hand" drawing of a population model can usually be enveloped by a linear fractional. (Cull [5] gives a discussion of dynamical systems defined by linear fractionals.) As we showed by example, enveloping by a linear fractional is only a *sufficient* condition for global stability. The simplest population models which have local stability without global stability are discussed by Singer [21] and by Cull [4]. While the examples in this paper are all one-humped population models, **enveloping implies global stability** also holds for functions with multiple peaks, for discontinuous functions, and even for multi-functions.

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