Linear Fractionals - Simple Models with Chaotic-like Behavior

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Abstract.
Since the seminal papers of Li and Yorke [5] and May [6] [7] the importance of *chaos* in science has been clear. But, the impressive theorems leading to an understanding of chaos have been largely proved for one-dimensional continuous maps and only conjectured to hold for more complicated systems. In this paper, we want to consider a simple but discontinuous family of maps, and ask how closely these maps, the linear fractionals, come to being chaotic. Our basic answers are that linear fractionals are easy to understand; that they do *not* show fully chaotic behavior in the technical sense; and that they do show behavior that would intuitively be considered chaotic. In particular, we show that many linear fractionals exhibit global asymptotic stability and do not show chaos; that rational coefficient linear fractionals may be periodic, but the the periods are restricted to be 1, 2, 3, 4, or 6; that more general periodicity is possible with irrational coefficients; that some linear fractionals are aperiodic but have chaotic-like orbits; that these aperiodic maps have invariant distributions, but such distributions are *not* attractive. We also show, following Cull and Chaffee [2], that linear fractionals may be used to show global stability of other nonlinear maps.

In summary, linear fractionals are easy-to-understand nonlinear maps that have a variety of applications, and these maps can display complex chaotic-like behavior.

Keywords: chaos, linear fractional functions, stability, maps of the interval, oscillations, population models, recurrences

1 INTRODUCTION

Since the seminal papers of Li and Yorke [5] and May [6] [7] the importance of *chaos* in science has been clear. But, the impressive theorems leading to an understanding of chaos have been largely proved for one-dimensional continuous maps and only conjectured to hold for more complicated systems. In this paper, we want to consider a simple but discontinuous family of maps, and ask how closely these maps, the linear fractionals, come to being chaotic. Our basic answers are that linear fractionals are easy to understand; that they do *not* show fully chaotic behavior in the technical sense; and that they do show behavior that would intuitively be considered chaotic. In particular, we show that many linear fractionals exhibit global asymptotic stability and do not show chaos; that rational coefficient linear fractionals may be periodic, but the the periods are restricted to be 1, 2, 3, 4, or 6; that more general periodicity is possible with irrational coefficients; that some linear fractionals are aperiodic but have chaotic-like orbits; that these aperiodic maps have invariant distributions, but such distributions are *not* attractive. We also show, following Cull and Chaffee [2], that linear fractionals may be used to show global stability of other nonlinear maps.

In summary, linear fractionals are easy-to-understand nonlinear maps that have a variety of applications, and these maps can display complex chaotic-like behavior.

The simplest nonlinear recurrences are the linear fractional recurrences, which are simply the ratio of two linear recurrences. These recurrences are worth studying because

1. they are simple
2. they will give examples of how nonlinear recurrences can work
3. techniques from linear recurrences can be used
4. linear fractions can be used to study other nonlinear recurrences.

We will write a linear fractional in the form

$$x_{t+1} = \frac{ax_t + b}{cx_t + d}$$

where we will assume that a, b, c, d are real constants and that x_t is a real variable. Later we will also consider the special case when the constants and variables are restricted to be rational numbers. Obviously, the linear fractionals are a special subclass of the general recurrence

$$x_{t+1} = f(x_t)$$

which is easy to analyze when f is a linear function but is impossible to analyze when f is a general function. By restricting f to the linear fractional form, we will be able to analyze and still have some nonlinear behavior. The usual questions about nonlinear systems include the existence of fixed points, the existence of cycles of various lengths, the asymptotic behavior of the system (i.e. are the fixed points or cycles attractive in some sense), local and global stability of fixed points and cycles, chaos or chaotic-like behavior, average behavior for a distribution of initial conditions. Pleasantly enough, all of these questions can be solved in a relatively easy fashion for linear fractionals.

We will address each of these problems in turn. First, we will look at the existence and stability of fixed points and cycles.

The behavior of a nonlinear system is usually analyzed in terms of the system's fixed points and cycles.

Definition 1 *For the system $x_{t+1} = f(x_t)$, the point p is a* **fixed point** *if and only if $p = f(p)$. The r-fold iteration of f is defined recursively by and $f^{(0)}(x) = x$ and $f^{(i+1)}(x) = f(f^{(i)}(x))$. A set of distinct points $p_1, ..., p_K$ forms a* **cycle of length** K *if and only if $f^{(K)}(p_i) = p_i$ for all $1 \le i \le K$. A fixed point p is* **attractive** *if and only if for some neighborhood of p and all points q in this neighborhood $\lim_{n\to\infty} f^{(n)}(q) = p$. Similarly, a cycle $p_1, ..., p_K$ is attractive if and only if for some neighborhood of the cycle and all points q in this neighborhood $\lim_{n\to\infty} f^{(nK)}(q) \in \{p_1, ..., p_K\}$.*

We would like to analyze linear fractions in terms of fixed points, cycles, and other nonlinear behavior. But, first we should note that a linear fraction can degenerate into a linear system when $c = 0$. That is,

$$x_{t+1} = \frac{\hat{a}x_t + \hat{b}}{0x_t + \hat{d}} = ax_t + b \tag{1}$$

The solution to such a linear recurrence is

$$x_t = a^t x_0 + b\sum_{i=0}^{t-1} a^i = \begin{cases} x_0 + bt & \text{if } a = 1 \\ a^t x_0 + b\frac{a^t - 1}{a-1} & \text{if } a \ne 1 \end{cases} \tag{2}$$

It is usual to analyze such a system in terms of the growth rate a. Generally, if $|a| > 1$ solutions will be increasing in absolute value while if $|a| < 1$ solutions will be decreasing in absolute value. In fact by setting $y = x + \frac{b}{a-1}$ the linear system (1) can be rewritten as $y_{t+1} = ay_t$ and the analysis in terms of growth rate seems perfectly appropriate. Of course, this transformation does not make sense when $a = 1$, since in the original (1) all solutions diverge while in the transformed system every point is a fixed point. If we analyze (1) as a nonlinear system we get a different picture. First, there is a fixed point at $-b/(a-1)$, and this point is attractive when $|a| < 1$ for all real x_0. To treat the increasing behavior when $|a| > 1$, we extend the real numbers by including ∞, the point at plus and minus infinity. Then the linear system will always have ∞ as a fixed point. This fixed point, ∞, will be attractive when

$|a| > 1$. Considering, this point when $a = 1$, also explains the discrepancy in the transformed system because when $a = 1$, each point is transformed to ∞ and is thus a fixed point. In summary, a linear system can be seen to have two fixed points one at $-b/(a-1)$ and the other at ∞. One of these will be attractive depending on whether $|a| < 1$ or $|a| > 1$. In the singular case when $a = 1$, the two fixed point degenerate into a single fixed point at ∞, which is an attractive fixed point. When one of two fixed points is attractive, the system exponentially converges toward the fixed point, but when there is only one fixed point the convergence is only linear.

Periodic behavior occurs when $a = -1$ then all points, except the two fixed points, have period 2. The period 2 cycles are not attractive. They are sometimes called *neutrally stable* because if the system starts in a point near a cycle, the system always stays near the cycle, but the system does not approach any nearer to the cycle. In the very special case when $a = 1$ and $b = 0$, every point becomes a neutrally stable fixed point. Finally, in the extremely singular case $a = 0$, all trajectories go to b in one step, and b is called a *superstable* point.

Behavior identical to the linear system (1) can also be found in the nonlinear system

$$z_{t+1} = \frac{z_t}{bz_t + a}.$$

The only difference is that x has been replaced by $1/x$, and so the fixed points have been shifted from $-b/(a-1)$ and ∞ to $(a-1)/-b$ and 0. This observation suggests that linear fractionals may have many features in common with linear systems, but that their analysis will depend on functions of several parameters and that these functions should be invariant to such obvious transformations as taking reciprocals of the system's variable.

2 ASYMPTOTIC BEHAVIOR

Let us now consider linear fractional systems in terms of what we have found about linear systems. For the fractional,

$$x_{t+1} = \frac{ax_t + b}{cx_t + d}$$

a key parameter is the determinant, $ad - bc$. We will assume that $c \neq 0$ to make the system nonlinear. Let us first consider linear fractionals with 0 determinant. Multiplying top and bottom of $f(x)$ by cd gives

$$f(x) = \frac{adcx + bcd}{cd(cx + d)} = \frac{bc(cx + d)}{cd(cx + d)} = \frac{b}{d}.$$

So, here after one iteration all trajectories go to b/d which is a superstable point. (The superstable point is a/c if $d = 0$.)

Next, consider the negative determinant case. A fixed point p will occur when $f(p) = p$ or equivalently when $0 = cp^2 + (d-a)p - b$. This quadratic equation will have two distinct real roots when the discriminant $(a-d)^2 + 4bc$ is positive. A linear analysis will show what happens in the long run. First, notice that a linear fractional can be represented in a linear fashion by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x_0 \\ 1 \end{pmatrix}$$

where x_0 is the initial point and the ratio of the components of the produced vector will be the value of the linear fractional. So n steps of the linear fractional can be computed by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^n \begin{pmatrix} x_0 \\ 1 \end{pmatrix}$$

and then taking a ratio of components. As usual, the powers of a matrix can be computed by diagonalizing the matrix and taking powers of the eigenvalues. The eigenvalues will be the two roots of $\lambda^2 - (a+d)\lambda +$

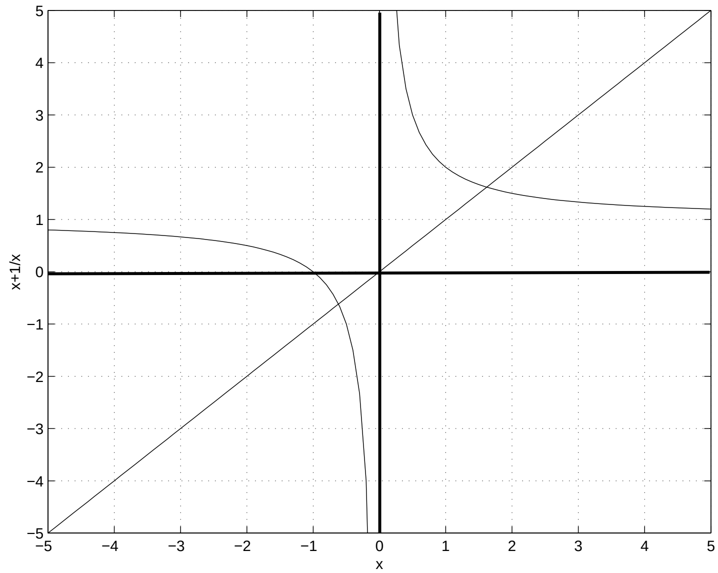

FIGURE 1. A plot showing a linear fractional with two fixed point. All iterates will converge to the upper fixed point.

$ad - bc$. Since, as above, the discriminant of this quadratic is positive, there will be two distinct real eigenvalues. The n^{th} term of the linear iteration can be written as

$$\frac{b}{\lambda_2 - \lambda_1} \left[\begin{array}{cc} 1 & 1 \\ \frac{\lambda_1 - a}{b} & \frac{\lambda_2 - a}{b} \end{array} \right] \left[\begin{array}{cc} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{array} \right] \left[\begin{array}{cc} \frac{\lambda_2 - a}{b} & 1 \\ -\frac{\lambda_1 - a}{b} & 1 \end{array} \right] \left(\begin{array}{c} x_0 \\ 1 \end{array} \right).$$

If we assume that $|\lambda_1| > |\lambda_2|$ then taking a ratio and a limit

$$\lim_{n\to\infty} x_n = \frac{b}{\lambda_1 - a}$$

unless $x_0 = \frac{b}{\lambda_2 - a}$. It is easy to check that these two points are in fact the fixed points, and so the linear fractional will have one unstable fixed point and one globally attractive fixed point when $|\lambda_1| \neq |\lambda_2|$. The given formulas are indeterminate when $b = 0$, but they can be written in the equivalent forms $\frac{\lambda_1 - d}{c}$ and $\frac{\lambda_2 - d}{c}$.

The more interesting case occurs when $|\lambda_1| = |\lambda_2|$. Then $a + d = 0$, and a simple calculation shows that $f(f(x)) = x$. So every point except for the two fixed points will have period 2. Of course, neither of the fixed points is attractive.

If the determinant is positive, the same analysis applies when $|\lambda_1| \neq |\lambda_2|$. That is, there are two fixed points; one is globally stable and the other is unstable. Here, the $|\lambda_1| \neq |\lambda_2|$ condition is equivalent to the discriminant, $(a - d)^2 + 4bc$, being positive.

When this discriminant is zero, there is only one fixed point. Geometrically, this corresponds to the line $y = x$ being tangent to the curve $y = f(x)$. The point of tangency is the fixed point. It is easy to see that every point above the fixed point will iterate to a point still above but nearer to the fixed point. While a point below the fixed point will iterate away from the fixed point, eventually one of the iterates will jump across the discontinuity and then jump to a point above the fixed point and then converge. As an example, consider $f(x) = x/(x + 1)$. Here, $x_n = \frac{x_0}{nx_0+1}$ and clearly every trajectory will converge to the fixed point at 0. The jumping between branches is hidden by this formula, but can be displayed by following, for example, the trajectory starting at $-3/4$ which gives $-3/4, -3, 3/4$ and then convergence down to 0. Notice that convergence is different in the one and two fixed point cases. For two fixed points, $x_n = F.P. + O(\gamma^n)$ where $F.P.$ is the stable fixed point and $|\gamma| < 1$, while in the one fixed point case $x_n = F.P. + O(1/n)$. Figure 1 shows the geometry of the two fixed point case and figure 2 shows the geometry of the one fixed point case.

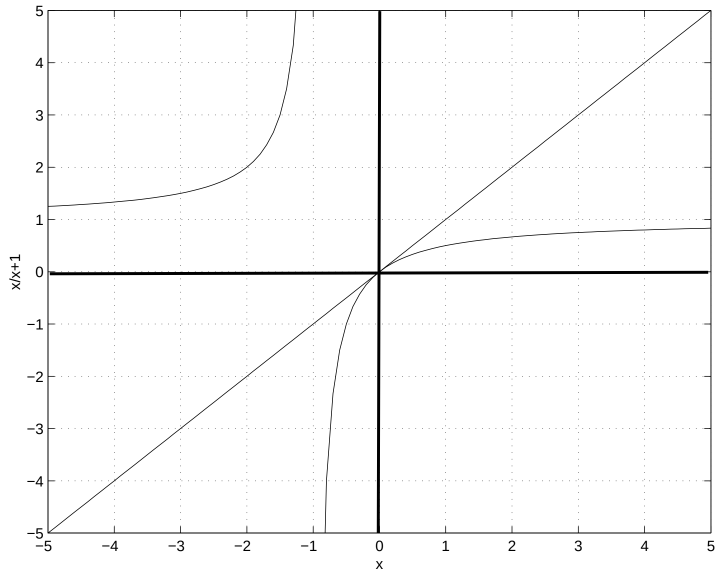

FIGURE 2. A plot showing a linear fractional with one fixed point. Convergence to the fixed point will be from above.

So far, linear fractionals have not behaved very differently from linear systems. The remaining cases in which the determinant is positive and the discriminant is negative will display more nonlinear behavior. We will discuss these cases in the following sections.

3 RATIONAL COEFFICIENTS AND PERIODICITY

Like other nonlinear systems, linear fractionals can display periodic behavior. But unlike other systems, linear fractionals do not allow the co-existence of cycles of different lengths.

Theorem 1 *For a linear fractional f, if there is a point x so that $f^{(K)}(x) = x$ then either x is a fixed point ($f(x) = x$) or for all y, $f^{(K)}(y) = y$ (all points have period K).*

Proof. The linear fractional $f(x) = \frac{ax+b}{cx+d}$ can be considered as a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ acting on a vector $\begin{pmatrix} x \\ 1 \end{pmatrix}$. Since A is 2×2, $A^K = \alpha_K A + \beta_K I$ for some scalars α_K and β_K, and $A^K \begin{pmatrix} x \\ 1 \end{pmatrix} = \alpha_K A \begin{pmatrix} x \\ 1 \end{pmatrix} + \beta_K \begin{pmatrix} x \\ 1 \end{pmatrix}$. So if $f^{(K)}(x) = x$,

$$\frac{\alpha_K(ax+b) + \beta_K x}{\alpha_K(cx+d) + \beta_K} = x.$$

There are two possibilities for this equation:

1. $\alpha_K = 0$ and so $A^K = \beta_K I$ and for every y, $A^K \begin{pmatrix} y \\ 1 \end{pmatrix} = \beta_K \begin{pmatrix} y \\ 1 \end{pmatrix} \equiv \frac{\beta_K y}{\beta_K} = y$, so all points have period K, or
2. $\alpha_K \neq 0$ and then $ax + b = (cx + d)x$ which is the fixed point equation.

□

This theorem allows for the co-existence of periodic points and fixed points. When the linear fractional has real eigenvalues, two real fixed points will occur, and periodic behavior will occur when the two eigenvalues have the same magnitude. In this case, all non-fixed points will have exact period 2.

When the eigenvalues are complex, there will be no real fixed points, and the magnitudes of the eigenvalues are forced to be equal. Here, let $e^{i\theta} = \lambda_1/\lambda_2$. If θ is a rational multiple of π, then there will be a least positive integer K, so that $A^K = I$. In this case, all points will have exact period K. If θ is not a rational multiple of π, the system will not be periodic and all points will fail to be periodic. We will consider this situation in more detail in the following section.

In general, all values of θ are possible, since a direct calculation of λ_1/λ_2 shows that any desired complex number of norm 1 can be produced by appropriate choice of the parameters a, b, c, d for the linear fractional. Of course, these parameters can be chosen as real numbers, but in general, these parameters may have to be irrational. In realistic uses of linear fractionals, one would like to assume that the parameters have finite representations. In particular, one might like to assume that these parameters are rational. Of course, if the parameters are rational, the same linear fractional can be represented using only integer parameters. We would like to know which periods are possible for rational parameter linear fractionals. The following theorem gives the answer.

Theorem 2 : *A linear fractional with rational (or integer) parameters can only have periods 1, 2, 3, 4, and 6, and there are linear fractionals with these periods.*

Proof. Later we will show by example that each of these periods do occur. Let $\gamma = \lambda_1/\lambda_2$. If the linear fractional has period K, then $\gamma^K = 1$. Now, γ is the root of a polynomial $\gamma^2 - e\gamma + 1$ where e is a rational function of a, b, c, d. Clearly, $\gamma^2 - e\gamma + 1$ must divide $\gamma^K - 1$. It can be shown that if K is the least such integer, then $\gamma^2 - e\gamma + 1$ must be a polynomial of degree $\phi(K)$ where $\phi(K)$ is the number of integers, i's, less than or equal to K which are relatively prime to K, i.e. $\gcd(i, K) = 1$. We argue that if $K > 6$, then $\phi(K) > 2$. Clearly, 1 and $K - 1$ are relatively prime to K. If there is a prime number p, so that $\sqrt{K} < p < K - 1$, then there is a third number relatively prime to K. But by Bertrand's postulate, for every $m > 1$, there is a prime between m and $2m$. So if $\sqrt{K} < (K-1)/2$, p will exist. But this inequality clearly holds for $K > 7$. So no periods larger than 7 are possible. It is easy to check that $\phi(1) = 1, \phi(2) = 1, \phi(3) = 2, \phi(4) = 2, \phi(6) = 2$, and that $\phi(5) = 4$ and $\phi(7) = 6$. So the only possible periods are $1, 2, 3, 4$, and 6. □

We now give examples of rational linear fractionals with the possible periods.

Period 1: $f(x) = x$. This is a degenerate linear fractional in which all points have period one.

Period 2: $f(x) = \frac{x-2}{2x-1}$. Here there are complex eigenvalues and no fixed points. An example period is $2 \longleftrightarrow 0$. For $f(x) = (5x-2)/(4x-5)$, there are real eigenvalues and two fixed points at $(5 \pm \sqrt{17})/4$. All other points have period 2, for example $1 \longleftrightarrow -3$.

Period 3: $f(x) = (x-1)/x$. Example period $3 \longrightarrow 2/3 \longrightarrow -1/2$

Period 4: $f(x) = (x-1)/(x+1)$. Example period $2 \longrightarrow 1/3 \longrightarrow -1/2 \longrightarrow -3$

Period 6: $f(x) = (2x-1)/(x+1)$. Example period $3 \longrightarrow 5/4 \longrightarrow 2/3 \longrightarrow 1/5 \longrightarrow -1/2 \longrightarrow -4$

A pleasant outcome of this analysis is that it is easy to test for periodicity in rational linear fractionals. One simply tries an initial condition and checks to see if it gives periodic behavior of length at most 6. The only minor problem is that one could chance on a fixed point and then one would need to test other points for periodicity.

4 CHAOTIC-LIKE BEHAVIOR

The standard definition of chaos due to Devaney [3] has three requirements:

1. cycles of every period

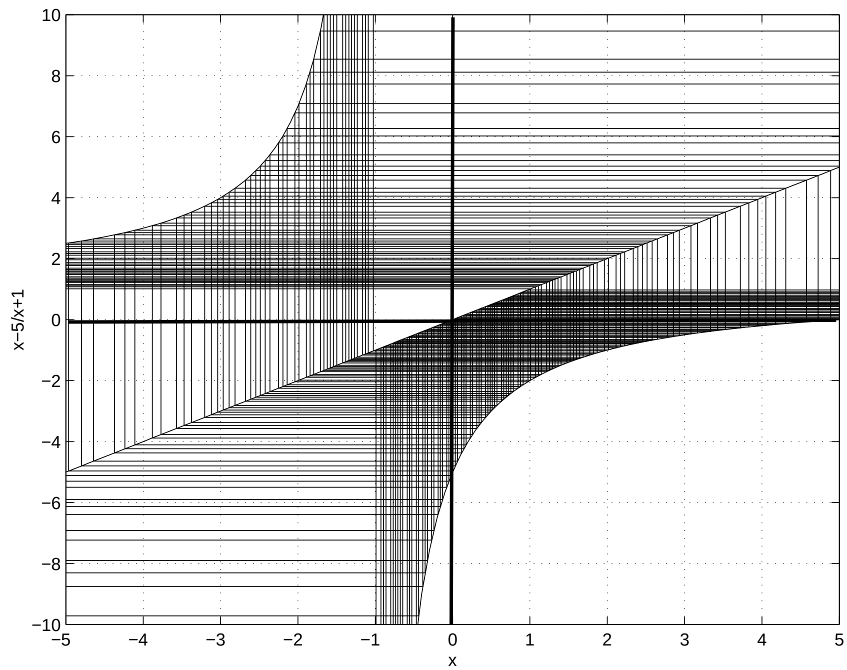

FIGURE 3. A Web Plot of Model x-5/x+1 showing that most points are visited

2. sensitive dependence on initial conditions

3. for every open set A and every open set B there is an $x_0 \in A$ and a $K \in N$ so that $f^{(K)}(x) \in B$.

For our linear fractionals, (1) is simply false, but (2) and (3) hold when the ratio of the eigenvalues does not correspond to a rational multiple of π. Sensitive dependence means that regardless of how closely two trajectories start, eventually these trajectories will be far apart. For linear fractionals, the pole at $p = -d/c$ will force trajectories to diverge from one another. For example, if one chose to consider trajectories starting at $p - \epsilon$ and $p + \epsilon$, then $|f(p + \epsilon) - f(p - \epsilon)| = 2|Det/c^2\epsilon|$ and this quantity can be made as large as one likes by taking ϵ small. For some linear fractionals, this will not be a problem because trajectories will be attracted toward stable fixed points and any initial divergence will disappear in the long term.

For periodic linear fractionals, initial difference will in essence be maintained and not increase. But as we will see, there are complex eigenvalues for which the dependence on initial conditions does not die out. Every trajectory will eventually come close to the pole and then be thrown far away. So two trajectories that start close together will eventually be far apart and there will be no tendency for them to again become close. To see what happens in this case, we will first consider (3) and show that it holds.

Definition 2 *A sequence $< S >$ is a* **source** *if for every α and for every $\epsilon > 0$ there is a K so that $|\alpha - S_K| < \epsilon$.*

Lemma 1 *If $< S >$ is a source and $g(z)$ is an onto function so that every pre-range element has a neighborhood of continuity then $g(< S >)$ is also a source.*

Proof. Let w be the desired point, then since g is onto there is a v so that $g(v) = w$. Since there is a neighborhood of continuity around v, if $|x - v| < \epsilon$ then $|g(x) - g(v)| = |g(x) - w| < \delta$. But since $< S >$ is a source, there is a K so that $|S_K - v| < \epsilon$ and so $|g(S_K) - w| < \delta$ for any desired w and δ, and so $g(< S >)$ is also a source. □

Theorem 3 *: If $< x_n >= x_0, x_1, ...$ is a trajectory of a linear fractional in which $\lambda_1/\lambda_2 = e^{i\theta}$ and θ is an irrational multiple of π, then $< x_n >$ is a source.*

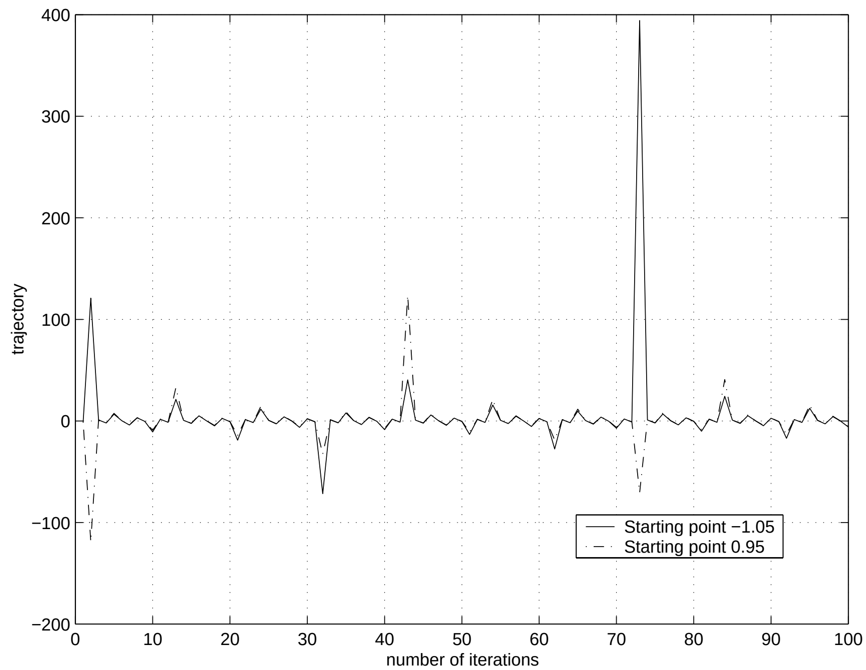

FIGURE 4. Two trajectories for $f(x) = \frac{x-5}{x+1}$. One trajectory starts at -1.05 and the other at -0.95. These two trajectories are often very close, but occasionally they are far apart.

Proof. : The n^{th} term of $< x_n >$ can be written as

$$x_n = \frac{-ax_0 - b + x_0[\cos\theta - \sin\theta\cot n\theta]}{a - cx_0 - [\cos\theta + \sin\theta\cot n\theta]}.$$

Since θ is an irrational multiple of π, $< n\theta > \bmod 2\pi$ is a source for $[0, 2\pi]$. But the transformation from $n\theta$ to x_n satisfies the hypotheses of the lemma, and so $< x_n >$ is a source for $(-\infty, \infty)$. □

Hence, these linear fractionals obey (3) and they also obey (2) because any two near-by trajectories will eventually hit a small neighborhood of the pole and then be thrown far apart. Figure 3 shows the web diagram for the linear fractional $(x-5)/(x+1)$. One can see that even with 100 iterates as used in the diagram calculation, almost all points are filled in, verifying that this fractional obeys (3). Figure 4 shows sensitive dependence on initial conditions in that the two displayed trajectories are often very close but occasionally they are far apart.

5 INVARIANT DISTRIBUTIONS

When a system has wandering trajectories that do not converge to cycles or fixed points, one asks if there are perhaps other properties that are conserved. The most obvious idea is to ask how long a trajectory stays in a region or how often a trajectory visits a region. It may be difficult to answer these questions and perhaps unenlightening because we want to know how the system behaves rather than how a specific trajectory behaves. So, we consider putting a probability distribution on the space and asking how the system changes this distribution. In particular, we would like to know if there is a very special distribution that remains the same after the system acts on it. If we could prepare enough copies of a system, and start these copies in accord with this invariant distribution, then if we looked at these copies later (even a long time later) we would still see the same distribution of states. Although the copies that started in particular states would no longer be in those states, other copies would be in those states in proportion to the invariant density. We could also hope for some sort of convergence . It might be possible that the invariant distribution was attractive if not for all initial distribution, then at least for a class of initial distribution. By attractive here we mean that under some reasonable definition of distance between distributions, the distance between the system's distribution at time t and the invariant distribution would go to 0, as t increases. As we will show linear fractionals do have invariant distributions but these distributions are not attractive.

Although we spoke of probability distribution, it may be easier to work with probability densities. A density function $g(x)$ defined on $(-\infty, \infty)$ will have the property that for every interval (α, β) the probability assigned to that interval is $\int_\alpha^\beta g(x)dx$. For normalization, we want $\int_{-\infty}^{\infty} g(x)dx = 1$. Now let us see how the mass on the interval (α, β) is transformed by the mapping f. First, the interval (α, β) is transformed to $(f(\alpha), f(\beta))$ and then the mass that was on (α, β) is spread out on this new interval. But if $g(x)$ is an invariant density, then the density on the new interval must also be $g(x)$. If we let $\hat{g}(x)$ be the distributive on the new interval, we have $\int_\alpha^\beta g(x)dx = \int_{f(\alpha)}^{f(\beta)} \hat{g}(x)dx$ and using $x = f(y)$ and $dx = f'(y)dy$ this becomes $\int_\alpha^\beta g(x)dx = \int_\alpha^\beta \hat{g}(x)f'(y)dy$. Since α and β are arbitrary, a limiting argument gives $g(x) = \hat{g}(f(x))f'(x)$ at least for those x's at which these functions are continuous. Said another way, $\hat{g}(x) = g(f^{-1}(x))/f'(f^{-1}(x))$. If $g_I(x)$ is an invariant density then $g_I(x) = g_I(f(x))f'(x)$. Since we want $g_I(x)$ to have a finite integral over $(-\infty, \infty)$, $g_I(x)$ should be going to 0 when x is large in absolute value, and so it may be easier to look at $h(x) = 1/g_I(x)$. Then $h(x) = h(f(x))/f'(x)$ and substituting a linear fractional for f, with the assumption that $ad - bc = 1$,

$$h(x) = (cx + d)^2 h(\frac{ax + b}{cx + d}).$$

Assuming that $h(x)$ has a power series, then $h(x)$ must be a polynomial of degree 2. The coefficients of the polynomial can be found by solving a set of linear equations. Up to an unknown scale factor, the unique solution to the set of three linear equations for $h(x)$ is

$$h(x) = cx^2 + (d - a)x - b.$$

Theorem 4 *If $f(x)$ is a linear fractional with complex eigenvalues, then there is a unique invariant density*

$$g(x) = \frac{\gamma}{cx^2 + (d - a)x - b},$$

where γ is determined from the normalization condition $\int_{-\infty}^{\infty} g(x)dx = 1$.

The complex eigenvalue condition is equivalent to $g(x)$ having no real poles which in turn implies that $g(x)$ is integrable and that the normalization makes sense.

Conveniently enough, the integral of the density function can also be written in closed form as

$$\int_{-\infty}^{z} g(x)dx = \frac{1}{\pi} \arctan \frac{2cz + d - a}{\sqrt{-(d - a)^2 - 4bc}} + \frac{1}{2}.$$

If one could choose the states of an ensemble of systems to satisfy the density $g_I(x)$, then as the states evolve by the application of f, the same density would be maintained. One might hope that if one chose any density then under f the density might evolve toward $g_I(x)$. The clue that this is not the case is in the theorem's condition that all one needs for an invariant distribution is that the eigenvalues are complex. But it is easy to construct a linear fractional with complex eigenvalues i and $-i$, and such a system will have all points periodic with period 2. For such a system, ANY initial distribution will repeat itself after two steps and the density will not approach the invariant density. More specifically, one can show the following theorem.

Theorem 5 *For a linear fractional with complex eigenvalues the invariant density is not attractive even within the class of densities whose reciprocals are polynomials of degree 2. For such densities, the discriminant is a conserved quantity.*

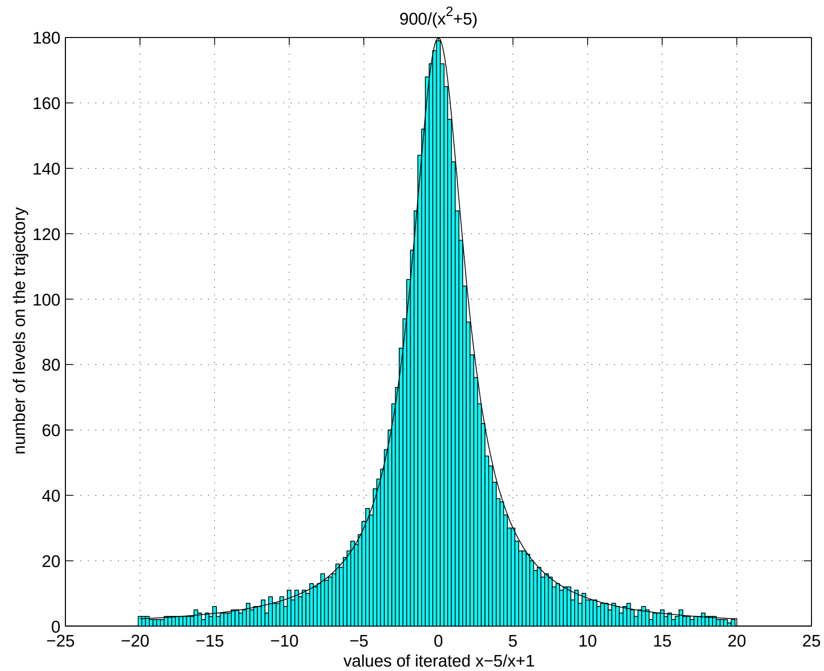

FIGURE 5. A histogram for a trajectory of $(x-5)/(x+1)$ showing good agreement with the density $1/(x^2+5)$.

The proof of this theorem consists of showing that the coefficients of the new density can be computed from the coefficients of the old density by applying a matrix to the vector of coefficients. The invariant density corresponds to the 1-eigenvector, the eigenvector associated with the eigenvalue 1. One can then show that the other two eigenvalues of the matrix also have absolute value equal to 1. Hence, even though a density will convert into a new density, the difference from the 1-eigenvector will not decrease and the 1-eigenvector corresponding to the invariant density will not be attractive. In fact, one can also show that the discriminant is preserved. Starting with a density which is the reciprocal of a degree 2 polynomial and iterating using f will result in a density of the same kind and the discriminant for this density will be identical to the discriminant for the starting density.

What will a typical aperiodic trajectory look like? One way to describe such a trajectory is to use a histogram, that is, break up the range into small bins and count the number of times the trajectory visits the bin. One can then normalize by the number of iterates, and hope that a limiting histogram exists.

Let $< f^{(n)}(x_0) >$ be the iterates of f starting at x_0. If this trajectory is well-behaved, then there is an associated histogram, H, so that $H((a,b))$ is the frequency with which the trajectory visits the interval (a,b), i.e.,

$$H((a,b)) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} I_{(a,b)}[f^{(i)}(x_0)]$$

where $I_{(a,b)}[z]$ is an indicator function which gives 1 when its argument is in (a,b) and 0 when its argument is not in (a,b). Assuming that these limits exist and that H is smooth, then

$$H(x) = \lim_{\epsilon\to 0} \lim_{N\to\infty} H((x-\epsilon, x+\epsilon))$$

should exist and behave like a probability density.

Assuming all of this is true, what should $H(x)$ look like? Since, we are looking at limiting behavior, it should not matter if the trajectory starts at x_0 or at $f(x_0)$. Hence, we can expect the limiting histogram to be a invariant density. But, by theorem 4 there is a unique invariant density, so $H(x)$ should look like $g_I(x)$. Figure 5 shows a histogram for 1000 iterates of $(x-5)/(x+1)$. This histogram looks quite smooth and agrees reasonably with the invariant density $1/(x^2+5)$.

These results are telling us that in spite of a seemingly irregular trajectory, a fairly simple property is maintained. If one looks at an individual trajectory, then the long-run histogram should look like a fairly simple function. On the other hand, if one took multiple copies of the same linear fractional system and assigned initial conditions for these with the probabilities given by the histogram, and then one looked at the distribution of states after one or several time intervals, the probability density will still be the same as the initial density. This is a sort of *ergodic* theorem which says that the average over one trajectory is the same as the appropriate average over an ensemble of systems.

Notice that if one picked a density $g(x)$ and picked the initial conditions for an ensemble according to $g(x)$, one would expect a different density say $g_1(x)$ to occur after one time step, and then densities $g_2(x), g_3(x), g_4(x), \ldots$ for subsequent time steps. There is NO reason to expect this sequence to converge toward a single density. In fact, theorem 5 says this sequence will not converge. But if one forms a sequence of averages, these averages will converge, i.e., let $G_0(x) = g(x)$, and $G_N(x) = \frac{1}{N}\sum_{i=0}^{N-1} g_i(x)$ then $\lim_{N\to\infty} G_N(x) = g_I(x)$.

6 SHOWING GLOBAL STABILITY

Our initial interest in linear fractionals came from an application to population models. For some years, it has been known that the usual one-dimensional population models were globally stable exactly when they were locally stable. [4] [1] [9] We were pleased to find that the usual population models were "enveloped" by linear fractionals, $\phi(x)$, of the special form:

$$\phi(x) = \frac{1-\alpha x}{\alpha - (2\alpha - 1)x} \qquad \text{where } \alpha \in [0,1) \ .$$

These special linear fractionals all have period 2, and so by a variation on Sarkovskii's theorem [8] we were able to show:

Theorem 6 *Let $\phi(x)$ be a monotone decreasing function which is positive on $(0, x_-)$ and so that $\phi(\phi(x)) = x$. Assume that $f(x)$ is a continuous function such that:*

- $\phi(x) > f(x)$ *on* $(0,1)$
- $\phi(x) < f(x)$ *on* $(1, x_-)$
- $f(x) > x$ *on* $(0,1)$
- $f(x) < x$ *on* $(1, \infty)$
- $f(x) > 0$ *on* $(1, x_\infty)$

then for all $x \in (0, x_\infty)$, $\lim_{k\to\infty} f^{(k)}(x) = 1$.

This theorem enabled us to show local stability implied global stability for the following seven population models:

- $x_{t+1} = x_t e^{r(1-x_t)}$
- $x_{t+1} = x_t[1 + r(1-x_t)]$
- $x_{t+1} = x_t[1 - r\ln x_t]$
- $x_{t+1} = x_t(\frac{1}{b+cx_t} - d)$
- $x_{t+1} = \frac{(1+ae^b)x}{1+ae^{bx}}$
- $x_{t+1} = \frac{(1+a)^b x}{(1+ax)^b}$ with $a > 0$, $b > 0$
- $x_{t+1} = \frac{rx}{1+(r-1)x^c}$

Details of these results appear in [2].

7 CONCLUSIONS

Linear fractionals form a simple class of nonlinear mappings. They display many of the behaviors possible for nonlinear systems. In particular, they exhibit both stable and unstable fixed points, periodic behavior, and even chaotic-like behavior. Table 1 gives a summary of the possible behaviors and the corresponding conditions on parameters.

TABLE 1. The possible behaviors for a linear fractional system $f(x) = \frac{ax+b}{cx+d}$ (we assume $c \neq 0$ to get a nonlinear system). $Det = ad - bc$. $Disc = (a-d)^2 + 4bc$.

$Det = 0$		$f(x) = constant$	Super stable fixed point
$Det < 0$ Two real fixed points	$d + a \neq 0$	One stable fixed point One unstable fixed point Convergence $x_n = F.P. + O(\lambda^n)$, $\|\lambda\| < 1$	
	$d + a = 0$	All points except the F.P.'s have period 2 neutral stability	
$Det > 0$	$Disc = 0$	One globally stable fixed point Convergence $x_n = F.P. + O(\frac{1}{n})$	
	$Disc > 0$	One stable fixed point One unstable fixed point Convergence $x_n = F.P. + O(\lambda^n)$, $\|\lambda\| < 1$	
	$Disc < 0$	Periodic, all points have the same period	Rational coeffs - possible periods 1,2,3,4,6
			Irrational coeffs - any period possible
		Chaotic-like	No periodic points All open sets visited Sensitive dependence on initial conditions Invariant density - Non-attractive

In contrast to to most nonlinear systems, linear fractionals can be analyzed using techniques from elementary mathematics and linear systems. Further, as we have shown, linear fractionals can be used to analyze more complicated systems. We suggest that linear fractionals should be a standard part of the toolbox for studying nonlinear systems.

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